

ON A REMARKABLE CLASS OF PARACONTACT METRIC MANIFOLDS

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ABSTRACT. We study a remarkable class of paracontact metric manifolds which have no contact metric counterpart: the paracontact metric $(-1, \tilde{\mu})$ -spaces which are not paraSasakian (i.e. have $\tilde{h} \neq 0$). We present explicit examples with \tilde{h} of every possible constant rank and some with non-constant rank, which were not known to exist until recently.

1. INTRODUCTION

Paracontact metric manifolds $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ have been studied by many authors in the recent years, particularly since the appearance of [14]. A special class among them is that of the $(\tilde{\kappa}, \tilde{\mu})$ -spaces, which satisfy [8]

$$(1) \quad R(X, Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

for all X, Y vector fields on M , where $\tilde{\kappa}$ and $\tilde{\mu}$ are constants and $\tilde{h} = \frac{1}{2}L_{\xi}\tilde{\varphi}$. These spaces include the paraSasakian manifolds [10, 14] and certain g -natural paracontact metric structures constructed on unit tangent sphere bundles [4], among others.

Although the nullity condition (1) seems very technical, paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -spaces appear naturally when studying the relation between contact and paracontact metric geometry. Indeed, any non-Sasakian contact metric (κ, μ) -space admits two paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -structures with the same contact form and, under some natural conditions, every non-paraSasakian paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -space accepts a contact metric (κ, μ) -structure with the same contact form [7, 8].

However, there are also some important differences between a contact metric (κ, μ) -space $(M, \varphi, \xi, \eta, g)$ and a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -space $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$. First of all, while they satisfy $h^2 = (\kappa - 1)\varphi^2$ and $\tilde{h}^2 = (\tilde{\kappa} + 1)\tilde{\varphi}^2$, respectively, the first condition implies that $\kappa \leq 1$ but the second one does not give any type of restriction over $\tilde{\kappa}$ because \tilde{g} is not positive definite [1, 8].

Another difference is that, in the contact metric case, $\kappa = 1$ is also equivalent to the manifold being Sasakian, i.e. $h^2 = 0$ implies $h = 0$. Nevertheless, there are non-paraSasakian paracontact metric $(-1, \tilde{\mu})$ -spaces, i.e. with $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$.

The first examples of these remarkable paracontact metric manifolds shown in the literature all had $\text{rank}(\tilde{h}) = n$ and $\tilde{\mu} = 0$ or 2, [5, 7, 8, 13]. Indeed, until very recently, there seemed to be no literature discussing the rank of \tilde{h} , if it had to be constant or why the values of μ zero and two were important.

Key words and phrases. paracontact metric manifold; contact metric manifold; (κ, μ) -spaces; nullity distribution; paraSasakian manifold.

This motivated the paper [11], where the author presented a local classification of paracontact metric $(-1, \tilde{\mu})$ -spaces in terms of the rank of \tilde{h} , examples of paracontact metric $(-1, 2)$ -spaces with every possible constant rank of \tilde{h} and an explanation of why the values of μ zero and two are special. Later, the author also wrote [12], where she gave an alternative proof of her main result, examples of paracontact metric $(-1, 0)$ -spaces with every possible constant rank of \tilde{h} and examples of paracontact metric $(-1, \tilde{\mu})$ -spaces where \tilde{h} is of non-constant rank.

In the present paper, after the preliminaries section, we will summarize what is known about the remarkable class of paracontact metric $(-1, \tilde{\mu})$ -spaces with $\tilde{h} \neq 0$, which have no contact metric counterpart.

2. PRELIMINARIES

Almost paracontact manifolds are $(2n + 1)$ -dimensional smooth manifolds endowed with a $(1, 1)$ -tensor $\tilde{\varphi}$, a vector field ξ and a 1-form η such that $\tilde{\varphi}^2 = I - \eta \otimes \xi$, $\eta(\xi) = 1$ and $\tilde{\varphi}$ induces a paracomplex structure on $\mathcal{D} = \ker \eta$, i.e. the eigendistributions \mathcal{D}^\pm corresponding to the eigenvalues ± 1 of $\tilde{\varphi}$ are both of dimension n [10, 14].

If the almost paracontact manifold admits a pseudo-Riemannian metric \tilde{g} of signature $(n + 1, n)$ such that $\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y)$ and $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for any vector fields X and Y , then M is called a *paracontact metric manifold* and $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ its paracontact metric structure, [14]. We refer to [3] for a recent survey on this type of manifold.

Given a paracontact metric manifold, the tensor field $\tilde{h} := \frac{1}{2}L_\xi \tilde{\varphi}$ is symmetric with respect to \tilde{g} , i.e. $\tilde{g}(\tilde{h}X, Y) = \tilde{g}(X, \tilde{h}Y)$, for all X, Y , and satisfies $\tilde{h}\tilde{\varphi} = -\tilde{\varphi}\tilde{h}$ and $\tilde{h}\xi = \text{tr } \tilde{h} = 0$ [14]. Moreover, $\tilde{h} = 0$ if and only if ξ is Killing, in which case the manifold is said to be a *K-paracontact manifold*.

An almost paracontact structure is called *normal* if and only if the tensor $[\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi = 0$, where $[\tilde{\varphi}, \tilde{\varphi}]$ is the Nijenhuis tensor of $\tilde{\varphi}$ [14]. A normal paracontact metric manifold is said to be a *paraSasakian manifold* and is in particular *K-paracontact*. The converse holds in dimension 3 [2] and always for $(-1, \tilde{\mu})$ -spaces [11, Th. 3.1]. However, it is not true in general, [11, Ex. 2.1].

Every paraSasakian manifold satisfies

$$(2) \quad R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y),$$

for every X, Y on M . The converse is not true, since Examples 3.8–3.11 of [11] and Examples 4.1 and 4.5 of [12] show that there are examples of paracontact metric manifolds satisfying Eq. (2) but with $\tilde{h} \neq 0$ (and therefore not K-paracontact or paraSasakian).

3. CLASSIFICATION AND EXAMPLES

Many examples of paraSasakian manifolds are known. For instance, hyperboloids $\mathbb{H}_{n+1}^{2n+1}(1)$ and the hyperbolic Heisenberg group $\mathcal{H}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}$, [9]. We can also obtain (η -Einstein) paraSasakian manifolds from contact metric (κ, μ) -spaces with $|1 - \frac{\mu}{2}| < \sqrt{1 - \kappa}$. In particular, the tangent sphere bundle T_1N of any space form $N(c)$ with $c < 0$ admits a canonical η -Einstein paraSasakian structure, [6]. Finally, we can see how to construct explicitly a paraSasakian structure on a Lie group, [11, Example 3.4], or directly on the unit tangent sphere bundle, [4].

On the other hand, until [11] and [12] only the following examples of paracontact metric $(-1, \mu)$ -spaces $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ with $\tilde{h} \neq 0$ were known (cited here in chronological order):

- paracontact metric $(-1, 2)$ -space with $\text{rank}(\tilde{h}) = n = 2$, [7].
- paracontact metric $(-1, 2)$ -spaces with $\text{rank}(\tilde{h}) = n = \text{arbitrary}$, [8].
- paracontact metric $(-1, 2)$ -space with $\text{rank}(\tilde{h}) = n = 1$, [13].
- paracontact metric $(-1, 0)$ -space with $\text{rank}(\tilde{h}) = n = 1$, [5].

We will first show why there only seem to be examples of paracontact metric $(-1, \tilde{\mu})$ -spaces with $\tilde{\mu} = 0$ or $\tilde{\mu} = 2$. Given a paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$, a \mathcal{D}_c -homothetic deformation is the following change [14]:

$$\tilde{\varphi}' := \tilde{\varphi}, \quad \xi' := \frac{1}{c}\xi, \quad \eta' := c\eta, \quad g' := c\tilde{g} + c(c-1)\eta \otimes \eta,$$

for some $c \neq 0$.

It is known that \mathcal{D}_c -homothetic deformations preserve the class of paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -spaces. In particular, if we deform a paracontact metric $(-1, \tilde{\mu})$ -space, we obtain another paracontact metric $(-1, \mu')$ -space with $\mu' = \frac{\tilde{\mu}-2+2c}{c}$.

Therefore, paracontact metric $(-1, 2)$ -spaces remain invariant under \mathcal{D}_c -homothetic deformations. Given a paracontact metric $(-1, 0)$ -space, if we \mathcal{D}_c -homothetically deform it with $c = \frac{2}{2-\tilde{\mu}} \neq 0$ for some $\tilde{\mu} \neq 2$, we will obtain a paracontact metric $(-1, \tilde{\mu})$ -space with $\tilde{\mu} \neq 2$. A sort of converse is also possible: given a $(-1, \tilde{\mu})$ -space with $\tilde{\mu} \neq 2$, a \mathcal{D}_c -homothetic deformation with $c = 1 - \frac{\tilde{\mu}}{2} \neq 0$ will give us a paracontact metric $(-1, 0)$ -space.

The case $\tilde{\mu} = 0$, $\tilde{h} \neq 0$ is special because the manifold satisfies (2) but it is not paraSasakian. Therefore, it makes sense to concentrate on $\tilde{\mu} = 0$ and $\tilde{\mu} = 2$.

We will now see that there are other possible ranks of \tilde{h} apart from n . We mention the following result, which appeared first in [11] and later with an alternative proof in [12].

Theorem 3.1 ([11, 12]). *Let M be a $(2n+1)$ -dimensional paracontact metric $(-1, \tilde{\mu})$ -space $(\tilde{\varphi}, \xi, \eta, \tilde{g})$. Then we have one of the following possibilities:*

- either $\tilde{h} = 0$ and M is paraSasakian,
- or $\tilde{h} \neq 0$ and $\text{rank}(\tilde{h}_p) \in \{1, \dots, n\}$ at every $p \in M$ where $\tilde{h}_p \neq 0$. Moreover, there exists a basis $\{\xi_p, X_1, Y_1, \dots, X_n, Y_n\}$ of $T_p(M)$ such that the only non-zero values of \tilde{g} on the basis are $\tilde{g}_p(\xi_p, \xi_p) = 1$ and $\tilde{g}_p(X_i, Y_i) = \pm 1$, and \tilde{h}_p can be written as either

$$\tilde{h}_{p|\langle X_i, Y_i \rangle} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \tilde{h}_{p|\langle X_i, Y_i \rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where obviously there are exactly $\text{rank}(\tilde{h}_p)$ submatrices of the first type.

If $n = 1$, such a basis $\{\xi_p, X_1, Y_1\}$ of $T_p(M)$ also satisfies that

$$\tilde{\varphi}_p X_1 = \pm X_1, \quad \tilde{\varphi}_p Y_1 = \mp Y_1.$$

Examples of paracontact metric $(-1, 2)$ -spaces with every possible constant rank of \tilde{h} were also presented in [11].

Example 3.2 $((2n + 1)$ -dimensional paracontact metric $(-1, 2)$ -space with $\text{rank}(\tilde{h}) = m \in \{1, \dots, n\}$, [11]). Let \mathfrak{g} be the $(2n + 1)$ -dimensional Lie algebra with basis $\{\xi, X_1, Y_1, \dots, X_n, Y_n\}$ such that the only non-zero Lie brackets are:

$$[\xi, X_i] = Y_i, \quad i = 1, \dots, m,$$

$$[X_i, Y_j] = \begin{cases} \delta_{ij}(2\xi + \sqrt{2}(1 + \delta_{im})Y_m) \\ \quad + (1 - \delta_{ij})\sqrt{2}(\delta_{im}Y_j + \delta_{jm}Y_i), & i, j = 1, \dots, m, \\ \delta_{ij}(2\xi + \sqrt{2}Y_i), & i, j = m + 1, \dots, n, \\ \sqrt{2}Y_i, & i = 1, \dots, m, j = m + 1, \dots, n. \end{cases}$$

If we denote G the Lie group whose Lie algebra is \mathfrak{g} , we can define a left-invariant paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ on G . Indeed, let us take the $(1, 1)$ -tensor $\tilde{\varphi}$ and the 1-form η such that

$$\begin{aligned} \tilde{\varphi}\xi &= 0, \quad \tilde{\varphi}X_i = X_i, \quad \tilde{\varphi}Y_i = -Y_i, \quad i = 1, \dots, n, \\ \eta(\xi) &= 1, \quad \eta(X_i) = \eta(Y_i) = 0, \quad i = 1, \dots, n. \end{aligned}$$

We define the metric \tilde{g} as the one whose only non-vanishing components are

$$\tilde{g}(\xi, \xi) = \tilde{g}(X_i, Y_i) = 1, \quad i = 1, \dots, n.$$

Long computations show that the manifold is a $(-1, 2)$ -space and that $\text{rank}(\tilde{h}) = m$.

Examples of $(2n + 1)$ -dimensional paracontact metric $(-1, 0)$ -spaces with $\text{rank}(\tilde{h}) = 1$ also appeared in [11] and were the first non-paraSasakian paracontact metric $(-1, \tilde{\mu})$ -spaces with $\tilde{\mu} \neq 2$ of dimension greater than 3 that were constructed. Later, examples of $(2n + 1)$ -dimensional paracontact metric $(-1, 0)$ -spaces with $\text{rank}(\tilde{h}) = m \in \{2, \dots, n\}$ were constructed by the author in [12].

Finally, the question of the existence of paracontact metric $(-1, \tilde{\mu})$ -spaces with \tilde{h} of non-constant rank was answered also in [12], where the author showed the first-known examples of paracontact metric $(-1, 2)$ -space and $(-1, 0)$ -space with $\text{rank}(\tilde{h}_p) = 0$ or 1 depending on the point $p \in M$. We show here one of them.

Example 3.3 (3) -dimensional paracontact metric $(-1, 2)$ -space with \tilde{h} of non-constant rank, [12]). Let us consider the manifold $M = \mathbb{R}^3$ with the usual cartesian coordinates (x, y, z) . The vector fields

$$e_1 = \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}$$

are linearly independent at each point of M . We can compute

$$[e_1, e_2] = 2\xi, \quad [e_1, \xi] = -x e_2, \quad [e_2, \xi] = 0.$$

We define the semi-Riemannian metric \tilde{g} as the non-degenerate one whose only non-vanishing components are $\tilde{g}(e_1, e_2) = \tilde{g}(\xi, \xi) = 1$, and the 1-form η as $\eta = 2ydx + dz$, which satisfies $\eta(e_1) = \eta(e_2) = 0$, $\eta(\xi) = 1$. Let $\tilde{\varphi}$ be the $(1, 1)$ -tensor field defined by $\tilde{\varphi}e_1 = e_1, \tilde{\varphi}e_2 = -e_2, \tilde{\varphi}\xi = 0$. Then $\Phi = d\eta$ and $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is a paracontact metric structure on M .

Moreover, $\tilde{h}\xi = 0$, $\tilde{h}e_1 = xe_2$, $\tilde{h}e_2 = 0$. Hence, $\tilde{h}^2 = 0$ and, given $p = (x, y, z) \in \mathbb{R}^3$, $\text{rank}(\tilde{h}_p) = 0$ if $x = 0$ and $\text{rank}(\tilde{h}_p) = 1$ if $x \neq 0$. Direct computations prove that the paracontact metric manifold M is also a $(-1, 2)$ -space.

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